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Operators: their Aluthge transforms and invariant subspaces

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This paper will be appeared in other journal. Let \mathcal{H} be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on \mathcal{H} . An arbitrary operator T in $\mathcal{L}(\mathcal{H})$ has a unique polar decomposition $T = UP$, where $P = (T^*T)^{\frac{1}{2}} = |T|$ and U is a partial isometry with initial space the closure of the range of $|T|$ and final space the closure of the range of T . Associated with T there is a related operator $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, sometimes called the Aluthge transform of T because it was studied in [1] in the context that T is a p -hyponormal operator (to be defined below). In this note we derive some spectral connections between an arbitrary $T \in \mathcal{L}(\mathcal{H})$ and its associated Aluthge transform \tilde{T} that enable us, in particular, to generalize an invariant-subspace-theorem of Berger [2] to that context. We will also show that the hyperinvariant subspace problems for hyponormal and p -hyponormal operators are equivalent.

The following lemma is completely elementary, but sets forth basic relations between T and \tilde{T} that will be useful throughout the paper.

Lemma 1.1. *Let $T = U|T|$ (polar decomposition) be an arbitrary operator in $\mathcal{L}(\mathcal{H})$ and let $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ be its Aluthge transform. Then*

$$(1) \quad |T|^{\frac{1}{2}}T = \tilde{T}|T|^{\frac{1}{2}},$$

and

$$(2) \quad T(U|T|^{\frac{1}{2}}) = (U|T|^{\frac{1}{2}})\tilde{T}.$$

In particular, T is a quasiaffinity (i.e., T is one-to-one and has dense range) if and only if $|T|$ is a quasiaffinity and U is a unitary operator, so \tilde{T} is a quasiaffinity if T is. Moreover, in this case, T and \tilde{T} are quasisimilar. Furthermore, T is invertible if and only if \tilde{T} is, and in this case, T and \tilde{T} are similar.

Remark 1.2. Consider the Hilbert space $\mathcal{H} = L^2([0, 1], \mu)$, where μ is Lebesgue measure, and let $\{e_n\}_{n=1}^{\infty}$ be any orthonormal basis for \mathcal{H} such that e_1 is the constant function 1. Let $U \in \mathcal{L}(\mathcal{H})$ be defined by $Ue_n = e_{n+1}$, $n \in \mathbf{N}$, so U is a unilateral shift, and consider $T = U(M_x)^2$, where M_x is multiplication by the position function. Then T is clearly not a quasiaffinity, but an easy calculation

shows that $\tilde{T} = M_x U M_x$ is a quasiaffinity. Thus the corresponding implication in Lemma 1.1 only goes one way.

Our first theorem shows that there is an intimate spectral connection between (an arbitrary operator) T and its associated \tilde{T} . As usual, we write $\sigma(T)$, $\sigma_p(T)$, and $\sigma_{ap}(T)$ for the spectrum, point spectrum, and approximate point spectrum of T , respectively.

Theorem 1.3. *For every T in $\mathcal{L}(\mathcal{H})$, $\sigma(T) = \sigma(\tilde{T})$, $\sigma_{ap}(T) = \sigma_{ap}(\tilde{T})$, $\sigma_p(T) = \sigma_p(\tilde{T})$, $\sigma_{ap}(T^*) \setminus (0) = \sigma_{ap}((\tilde{T})^*) \setminus (0)$, and $\sigma_p(T^*) \setminus (0) = \sigma_p((\tilde{T})^*) \setminus (0)$.*

We remark here that the example given in Remark 1.2 shows that all the spectral equalities in Theorem 1.3 are best possible.

For an operator $A \in \mathcal{L}(\mathcal{H})$, we write, as usual, $\sigma_e(A)$, $\sigma_{le}(A)$, and $\sigma_{re}(A)$ for the essential (Calkin), left essential, and right essential spectra of A , respectively. Recall that $\lambda \in \sigma_{le}(A)$ if and only if there exists an orthonormal sequence $\{e_n\}$ in \mathcal{H} such that $\lim_{n \rightarrow \infty} \|(A - \lambda)e_n\| = 0$, or, equivalently, if and only if there exists a sequence $\{f_n\}$ of unit vectors in \mathcal{H} such that $\{f_n\}$ converges weakly to zero and $\lim_{n \rightarrow \infty} \|(A - \lambda)f_n\| = 0$.

Corollary 1.4. *For any $T \in \mathcal{L}(\mathcal{H})$ with associated Aluthge transform \tilde{T} , we have $\sigma_e(T) = \sigma_e(\tilde{T})$, $\sigma_{le}(T) = \sigma_{le}(\tilde{T})$, and $\sigma_{re}(T) \setminus (0) = \sigma_{re}(\tilde{T}) \setminus (0)$.*

We turn now to the intimate connection between the invariant subspace lattices of an arbitrary operator T and its associated \tilde{T} . As usual, we write $\text{Lat}(A)$ for the invariant subspace lattice of an arbitrary operator $A \in \mathcal{L}(\mathcal{H})$. If $T \in \mathcal{L}(\mathcal{H})$ is not a quasiaffinity, then $0 \in \sigma_p(T) \cup \sigma_p(T^*)$, so trivially T has a nontrivial invariant subspace. Thus when investigating the relation between $\text{Lat}(T)$ and $\text{Lat}(\tilde{T})$, it suffices to consider the case that T is a quasiaffinity.

The following is an improvement of [5, Theorem 2].

Theorem 1.5. *Let $T = U|T|$ (polar decomposition) be an arbitrary quasiaffinity in $\mathcal{L}(\mathcal{H})$. Then the mapping*

$$\phi : \mathcal{N} \longrightarrow (|T|^{\frac{1}{2}}\mathcal{N})^-, \quad \mathcal{N} \in \text{Lat}(T),$$

maps $\text{Lat}(T)$ into $\text{Lat}(\tilde{T})$, and moreover if $(0) \neq \mathcal{N} \neq \mathcal{H}$, then

$$(0) \neq \phi(\mathcal{N}) = (|T|^{\frac{1}{2}}\mathcal{N})^- \neq \mathcal{H}.$$

Moreover the mapping

$$\psi : \mathcal{M} \longrightarrow (U|T|^{\frac{1}{2}}\mathcal{M})^-, \quad \mathcal{M} \in \text{Lat}(\tilde{T}),$$

maps $\text{Lat}(\tilde{T})$ into $\text{Lat}(T)$, and if $(0) \neq \mathcal{M} \neq \mathcal{H}$, then

$$(0) \neq \psi(\mathcal{M}) = (U|T|^{\frac{1}{2}}\mathcal{M})^- \neq \mathcal{H}.$$

Consequently, $\text{Lat}(T)$ is nontrivial if and only if $\text{Lat}(\tilde{T})$ is nontrivial.

Remark 1.6. If T in Theorem 1.5 is invertible, then T and \tilde{T} are similar (see Lemma 1.1), and thus have isomorphic invariant subspace lattices. Whether this is true for an arbitrary noninvertible quasiaffinity T , the authors have not been able to determine. Note that Theorem 1.5 implies that if one is trying to solve the invariant subspace problem for a particular quasiaffinity $T \in \mathcal{L}(\mathcal{H})$, it suffices to show that \tilde{T} has a nontrivial invariant subspace.

Definition 1.7 [1,6]. Suppose $T \in \mathcal{L}(\mathcal{H})$ and satisfies $(T^*T)^p \geq (TT^*)^p$ for some p in the interval $(0, +\infty)$. Then T is called a p -hyponormal operator. If $p = \frac{1}{2}$, T is sometimes called semi-hyponormal [1] and if $p = 1$, T is hyponormal.

There is a vast literature concerning p -hyponormal operators (for $0 < p < 1$) in which various special cases of Theorem 1.3 are proved (for p -hyponormal operators). For our purposes, we need only the following consequences of Löwner's inequality [8].

Remark 1.8. If $T \in \mathcal{L}(\mathcal{H})$ is a p -hyponormal operator for some p in the interval $(0, +\infty)$, then T is also q -hyponormal for every $0 < q \leq p$. In particular, an operator that is a p -hyponormal operator for some $p > 1$ is also hyponormal. Thus the interest in p -hyponormal operators has been concentrated mainly (but not exclusively; cf., for example, [4]) on those p -hyponormal operators for which $0 < p \leq 1$.

The following lemma was proved in [1] in the special case in which the partial isometry U in the polar decomposition $T = U|T|$ is a unitary operator, but the proof carries over to the general case. We introduce the proof here for conveniences.

Lemma 1.9([1]). Suppose that $T = U|T|$ (polar decomposition) is an arbitrary p -hyponormal operator in $\mathcal{L}(\mathcal{H})$ for some $p \in [\frac{1}{2}, 1]$. Then its Aluthge transform \tilde{T} is a hyponormal operator.

The following lemma was also proved in [1] in case $T = U|T|$ with U a unitary operator, but once again, the proof can be made to work in general.

Lemma 1.10([1]). Suppose $T = U|T|$ (polar decomposition) is an arbitrary p -hyponormal operator in $\mathcal{L}(\mathcal{H})$ for some p in the interval $(0, \frac{1}{2})$. Then \tilde{T} is a $(p + \frac{1}{2})$ -hyponormal operator and the Aluthge transform $\tilde{\tilde{T}}$ of \tilde{T} is a hyponormal operator.

If \mathcal{U} is a bounded open set in the complex plane \mathbb{C} , recall that a subset $\Lambda \subset \mathcal{U}$ is said to be *dominating* for \mathcal{U} if every bounded function $h(z)$ holomorphic on \mathcal{U} satisfies

$$\sup_{z \in \mathcal{U}} |h(z)| = \sup_{z \in \mathcal{U} \cap \Lambda} |h(z)|.$$

First we recapture the following corollary.

Corollary 1.11[5, Theorem 3] *Suppose $T = UP$ (polar decomposition) is an arbitrary p -hyponormal operator for some $p \in (0, +\infty)$, and suppose that there exists a nonempty open set \mathcal{U} in \mathbb{C} such that $\sigma(T) \cap \mathcal{U}$ is dominating for \mathcal{U} . Then T has a nontrivial invariant subspace.*

The following theorem generalizes a surprising theorem of Berger [2] for hyponormal operators to the context of p -hyponormal operators.

Theorem 1.12. *Let $T \in \mathcal{L}(\mathcal{H})$ be an arbitrary p -hyponormal operator for some $p \in (0, +\infty)$. Then there exists a positive integer K such that for all positive integers $k \geq K$, T^k has a nontrivial invariant subspace.*

Recall that a subspace \mathcal{M} of \mathcal{H} is a nontrivial hyperinvariant subspace for an operator $T \in \mathcal{L}(\mathcal{H})$ if $(0) \neq \mathcal{M} \neq \mathcal{H}$ and \mathcal{M} is invariant under every operator in the commutant $\{S \in \mathcal{L}(\mathcal{H}) : ST = TS\}$ of T . We write $\text{Hlat}(A)$ for the lattice of hyperinvariant subspaces of an operator $A \in \mathcal{L}(\mathcal{H})$. If $0 \neq T \in \mathcal{L}(\mathcal{H})$ and T is not a quasiaffinity, then $0 \in \sigma_p(T) \cup \sigma_p(T^*)$ and $\text{Hlat}(T) \neq \{(0), \mathcal{H}\}$ for trivial reasons. Then when investigating the relation between $\text{Hlat}(T)$ and $\text{Hlat}(\tilde{T})$, it suffices to that the case in which T is a quasiaffinity.

Theorem 1.13 *Let $T \in \mathcal{L}(\mathcal{H})$ be an arbitrary nonzero quasiaffinity. Then T has a nontrivial hyperinvariant subspace if and only if its Aluthge transform \tilde{T} does. Thus the hyperinvariant subspace problem for p -hyponormal operators (any $p \in (0, +\infty)$) is equivalent to the hyperinvariant subspace problem for hyponormal operators.*

Recall that $T \in \mathcal{L}(\mathcal{H})$ is a log-hyponormal operator if T is invertible and $\log(TT^*) \leq \log(T^*T)$. Note that any invertible p -hyponormal operator is log-hyponormal.

Theorem 1.14 *Suppose $T = UP$ (polar decomposition) is an arbitrary log-hyponormal operator, and suppose that there exists a nonempty open set \mathcal{U} in \mathbb{C} such that $\sigma(T) \cap \mathcal{U}$ is dominating for \mathcal{U} . Then T has a nontrivial invariant subspace.*

Recall that $T \in \mathcal{L}(\mathcal{H})$ is an ∞ -hyponormal operator if T is n -hyponormal for any natural number n . Note that any ∞ -hyponormal operator is p -hyponormal for any positive real number p .

We now close the paper as the following problem.

Problem 1.15 *Suppose T is an arbitrary ∞ -hyponormal operator. Does T have a nontrivial invariant subspace.*

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